UGEB2530 Games and Strategic Thinking
Sequential games

1. There is a number of chips on the table. Two players play a game with the following rules. In each turn, a player may remove 1,3,4 or 5 chips from the table. The player who makes the last move wins.

   (a) Determine whether n is a P-position or an N-position for $n = 8, 9, 10, 11$.

   (b) Find a winning move for the first player if initially there are n chips for $n = 100, 101, 102$.

Solution.

(a)

0	1	2	3	4	5	6	7	8	9	10	11	12
P	N	P	N	N	N	N	N	P	N	P	N	N

We see that $8, 10$ are P-position and $9, 11$ are N-position.

(b) Position n is a P-position if the remainder is 0 or 2 when n is divided by 8. Thus we have the following winning moves.

n	100	101	102
Winning moves	96	96,98	98

2. Let $\oplus$ denotes the nim-sum.

   (a) Find $27 \oplus 17$

   (b) Find x if $x \oplus 38 = 25$.

   (c) Prove that if $x \oplus y \oplus z = 0$, then $x = y \oplus z$.

Solution.

(a) Write the numbers in binary form $27 = 11011_2$ and $17 = 10001_2$. Now

$$\begin{array}{rrrrrr} & 1 & 1 & 0 & 1 & 1_2 \\ \oplus & 1 & 0 & 0 & 0 & 1_2 \\ \hline & & 1 & 0 & 1 & 0_2 \end{array}$$

Thus $27 \oplus 17 = 1010_2 = 10$.

(b)

$$\begin{aligned} x \oplus 38 &= 25 \\ x \oplus 38 \oplus 38 &= 25 \oplus 38 \\ x &= 25 \oplus 38 \\ &= 63 \end{aligned}$$

(c)

$$\begin{aligned} & x \oplus y \oplus z &= 0 \\ \Rightarrow & x \oplus y \oplus z \oplus y \oplus z &= y \oplus z \\ \Rightarrow & x \oplus (y \oplus y) \oplus (z \oplus z) &= y \oplus z \\ \Rightarrow & x \oplus 0 \oplus 0 &= y \oplus z \\ \Rightarrow & x &= y \oplus z \end{aligned}$$

3. Find all winning moves in the game of nim,

   (a) with three piles of 12, 19, and 27 chips.

   (b) with four piles of 13, 17, 19, and 23 chips.

Solution.

(a) Write the numbers in binary form $12 = 1100_2$, $19 = 10011_2$ and $27 = 11011_2$. Consider the nim-sum

$$\begin{array}{cccccc} & & 1 & 1 & 0 & 0_2 \\ & 1 & 0 & 0 & 1 & 1_2 \\ \oplus & 1 & 1 & 0 & 1 & 1_2 \\ \hline \\ & & & 1 & 0 & 0_2 \end{array}$$

We see that $(12, 19, 27)$ has a winning move to $(8, 19, 27)$.

(b) For the nim game with four pile of chips, a position (a, b, c, d) is a P-position if $a \oplus b \oplus c \oplus d = 0$. Write the numbers in binary form $13 = 1101_2$, $17 = 10001_2$, $19 = 10011_2$ and $23 = 10111_2$. Consider the nim-sum

$$\begin{array}{cccccc} & & 1 & 1 & 0 & 1_2 \\ & 1 & 0 & 0 & 0 & 1_2 \\ & 1 & 0 & 0 & 1 & 1_2 \\ \oplus & 1 & 0 & 1 & 1 & 1_2 \\ \hline \\ & 1 & 1 & 0 & 0 & 0_2 \end{array}$$

The position $(13, 17, 19, 23)$ has three winning moves to $(13, 9, 19, 23)$, $(13, 17, 11, 23)$ and $(13, 17, 19, 15)$.